ON A REFLECTIVE SUBCATEGORY OF THE CATEGORY OF ALL TOPOLOGICAL SPACES

BY

LADISLAV SKULA

1. **Introduction.** In his paper [4] J. F. Kennison gives three types of full reflective subcategories of the category of all topological spaces called simple, identifying and embedding and notes that he does not know whether these three types include all of the full reflective subcategories of all topological spaces.

In my paper I describe another type called *b-embedding* not mentioned in [4] and I prove that *if P is a full reflective* (*replete*) *subcategory of the category of all topological spaces containing at least one non-T_1-space, then P is a subcategory of one of the above-mentioned types* (*Theorem 3.5*).

For the understanding of my paper the basic knowledge of topology according to Kelley's *General topology* [3] is required. Thus a topological space will be an ordered pair (A, L), where L is the family of all open subsets of the underlying set A. (If there is no danger of confusion only A will denote the topological space.) The closure of $M \subset A$ will be denoted by $\mathrm{cl}_{(A,L)} M = \mathrm{cl}_A M = \mathrm{cl}\, M$; T will mean the category of all topological spaces with morphisms as continuous maps, and T_0 (resp. T_1) the full subcategory of T_0-spaces (resp. T_1-spaces). The topological product of topological spaces X_ι, $\iota \in \mathscr{I} \neq \varnothing$ will be denoted by $\prod X_\iota$ $(\iota \in \mathscr{I})$.

Further I presuppose the knowledge of Kennison's paper [4]. Recall that if F is a reflector of a category A in a category B, then for $X \in A$ e_X will denote the *reflection map* or the *front adjunction map* ($\in \mathrm{Hom}(X, F(X))$) with the property $e_Y f = F(f) e_X$ for all $f \in \mathrm{Hom}(X, Y)$. By a *topological property* there is meant a full subcategory of T which is closed under the formation of equivalent (=homeomorphic) objects (full replete subcategory of T). A *productive* (resp. a *closed hereditary*) *topological property* is such that it is closed under the formation of product spaces (resp. closed subspaces). The full subcategory P of T is *simple* (resp. *identifying*, or *embedding*) iff there exists a reflector $F: T \to P$ such that $e_X: X \to F(X)$ is one-to-one and onto for all $X \in T$ (resp. e_X is onto, or P contains only Hausdorff spaces and e_X is onto a dense subset).

From the Theorem on p. 878 in Notes [1] follows immediately:

1.1. *Let P be a topological property which is reflective in T. Then P is productive.*

2. ***b*-topology.**

2.1. Let $(A, L) \in T$. For $X \subset A$ let $bX = b_A X = b_{(A,L)} X$ denote the set of all points

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$p \in A$ with the following property: there exist no $G_1, G_2 \in L$ such that $p \in G_2 - G_1$ and $G_2 \cap X = G_1 \cap X$. For $X, Y \subset A$ the following assertion holds:

(a) $X \subset bX \subset \text{cl } X$,

(b) $X \subset Y \Rightarrow bX \subset bY$,

(c) $bX = bbX$,

(d) $b(X \cup Y) = bX \cup bY$.

Proof. The assertions (a) and (b) are clear. Assertion (c) follows in the following manner: according to (a) and (b) there holds $bX \subset bbX$. Let $p \in A - bX$. Then there exist $G_1, G_2 \in L$, $p \in G_2 - G_1$, $G_2 \cap X = G_1 \cap X$. Hence for $x \in (G_2 - G_1) \cup (G_1 - G_2)$ there holds $x \notin bX$. Thus $G_2 \cap bX = G_1 \cap bX$, therefore $p \notin bbX$.

Assertion (d): There holds $X \subset X \cup Y$, $Y \subset X \cup Y$, whence according to (b), $bX \cup bY \subset b(X \cup Y)$. Let $p \in A - bX \cup bY$; then there exist $G_1, G_2, G_1', G_2' \in L$ such that $G_1 \cap X = G_2 \cap X$, $G_1' \cap Y = G_2' \cap Y$, $p \in G_2 - G_1$ and $p \in G_2' - G_1'$. Put $H_2 = G_2 \cap G_2'$, $H_1 = (G_1 \cup G_1') \cap H_2$. Then $H_1, H_2 \in L$, $H_1 \cap (X \cup Y) = H_2 \cap (X \cup Y)$, $p \in H_2 - H_1$. Hence $p \notin b(X \cup Y)$.

On the space A a new topology, which we shall call *b-topology* (*of the topological space* A), is therefore defined as a closure by means of the operator b. To the topological concepts belonging to this b-topology we shall give the usual names preceded by b, e.g. b-closed set, b-dense set, etc.

2.2. *Let* $(A, L) \in T$, $\mathfrak{A} = \{(G_2 - G_1) \cup (G_1 - G_2) \mid G_1, G_2 \in L\}$. *Then* $\mathfrak{A}$ *is a base of open sets with regard to the b-topology of the space* (A, L).

Proof. I. Let $G_1, G_2 \in L$, $M = A - (G_2 - G_1) \cup (G_1 - G_2)$. Then $G_2 \cap M = G_1 \cap M$, whence $bM = M$, thus $(G_2 - G_1) \cup (G_1 - G_2)$ is a b-open set.

II. Let $x \in A$, U be a b-open set and $x \in U$. Then $x \notin b(A - U)$, hence there exist $G_1, G_2 \in L$, $x \in G_2 - G_1$, $G_2 \cap (A - U) = G_1 \cap (A - U)$. Then $[(G_2 - G_1) \cup (G_1 - G_2)] \cap b(A - U) = \varnothing$, whence $(G_2 - G_1) \cup (G_1 - G_2) \subset U$. Q.E.D.

COROLLARY. *Let* $G \in L$. *Then* $bG = G$.

Proof. Putting $G_1 = G$, $G_2 = A$, then according to 2.2, $A - G$ is b-open, therefore G is b-closed.

REMARK. The base $\mathfrak{A}$ however is not generally the system of all b-open sets of the space (A, L).

For example: let A be the set of all positive integers, for $n \in A$ put $X_n = \{m \in A \mid m \geqq n\}$, $L = \{X_n \mid n \in A\} \cup \{\varnothing\}$. Then $(A, L) \in T$ and the b-topology of the space (A, L) is discrete. But $\mathfrak{A} = L \cup \{X_n - X_m \mid n, m \in A\} \neq 2^A$.(1)

2.3. DEFINITION. A topological space B is called a *b-hull of a space* A if A is a subspace of B and $bA = B$; in other words A is b-dense in B.

2.4. *Let* $(B, K) \in T_0$ *be a b-hull of a space* (A, L). *Then* card $B \leqq$ exp exp card A.(2)

Proof. Let $b_1, b_2 \in B$, $b_1 \neq b_2$. Put $\mathfrak{A}_i = \{U \cap A \mid U$ is a neighborhood of the point $b_i\}$ for $i = 1, 2$. Then $\mathfrak{A}_1, \mathfrak{A}_2$ are filters on the set A. We can suppose that there

(1) 2^A denotes the system of all subsets of the set A.

(2) The symbol card M denotes the cardinal number of the set M and exp card M = card 2^M.

exists $U \in K$, $b_2 \in U$, $b_1 \notin U$. It holds that $U \cap A \in \mathfrak{A}_2$. If $U \cap A \in \mathfrak{A}_1$, then there exists $V \in K$, $b_1 \in V$, $V \cap A \subset U \cap A$. Put $V \cup U = W$. Then $W \in K$, $W \cap A = U \cap A$, $b_1 \in W - U$, therefore $b_1 \notin bA$, which is a contradiction.

Hence $U \cap A \in \mathfrak{A}_1$, therefore $\mathfrak{A}_1 \neq \mathfrak{A}_2$. Then card $B \leqq$ card $F(A)$, where $F(A)$ is the set of all filters on the set A. Since card $F(A) = \exp \exp$ card A, it holds that card $B \leqq \exp \exp$ card A.

2.5. *Let $A, B \in T$, $C \in T_0$, B be a b-hull of A, $f \in \mathrm{Hom}_T(A, C)$. Then there exists at most one continuous extension of f on B. If C is an arbitrary topological space and f^* any continuous extension of f on B, then $f^*(B) \subset bf(A)$.*

Proof. I. Let g, h be two different continuous extensions of f on B. Then there exists $y \in B$ such that $g(y) \neq h(y)$. We can suppose that there exists an open set G' of the space C such that $g(y) \in G'$, $h(y) \in G'$. Put $G_1 = h^{-1}(G')$, $G_2 = g^{-1}(G')$. Then G_1, G_2 are open sets of the space B and $G_1 \cap A = G_2 \cap A$, $y \in G_2 - G_1$, which is a contradiction.

II. Let f^* be the continuous extension of f on B, $x \in B$, G_1', G_2' be open sets of the space C, $G_1' \cap f(A) = G_2' \cap f(A)$, $f^*(x) \in G_2' - G_1'$. Then $G_1 = f^{*-1}(G_1')$, $G_2 = f^{*-1}(G_2')$ are open sets of the space B, $G_1 \cap A = G_2 \cap A$ and $x \in G_2 - G_1$, which is a contradiction. Thus $f^*(B) \subset bf(A)$.

REMARK. After writing this paper I learned that S. Baron obtained the first part of 2.5 in [5]. S. Baron defined equivalently $p \in bX$ iff $p \in \mathrm{cl}\,(X \cap \mathrm{cl}\,\{p\})$.

2.6 *Let $X, Y \in T_0$, Y be a b-hull of X, $f \in \mathrm{Hom}_T(Y, X)$, $f \mid X = i_X$.*[3] *Then $Y = X$.*

Proof. Let $j: X \to Y$ be the identity embedding of X into Y. Then $fj = i_X$. Clearly $jfj = j$ and $i_Y j = j$ so jf and i_Y are continuous extensions of j on Y. According to 2.5 $jf = i_Y$, which implies $X = Y$.

2.7. *Let $\mathscr{I} \neq \varnothing$ be a set, $X_\iota \in T$ and $M_\iota \subset X_\iota$ be a b-closed set for each $\iota \in \mathscr{I}$, $X = \prod X_\iota$ $(\iota \in \mathscr{I})$, $M = \{x \in X \mid x(\iota) \in M_\iota$ for each $\iota \in \mathscr{I}\}$. Then M is b-closed in X.*

Proof. Let p_ι be the ιth projection map $X \to X_\iota$ for all $\iota \in \mathscr{I}$. According to 2.5 $p_\iota bM \subset bp_\iota(M) = bM_\iota = M_\iota$. This implies $bM \subset M$ and so M is b-closed.

2.8. *Let $X, Y \in T$, $Z \in T_0$, $Z \notin T_1$, $f \in \mathrm{Hom}_T(X, Y)$. Let $\phi = \psi$ for each ϕ, $\psi \in \mathrm{Hom}_T(Y, Z)$, $\phi f = \psi f$. Then $bf(X) = Y$.*

Proof. Let $y \in Y - bf(X)$. Then there exist open sets G_1, G_2 of the space Y such that $G_1 \cap f(X) = G_2 \cap f(X)$ and $y \in G_2 - G_1$. According to the suppositions mentioned in the assertion there exist $z_1, z_2 \in Z$, $z_1 \in \mathrm{cl}\,\{z_2\}$, $z_2 \notin \mathrm{cl}\,\{z_1\}$. For $i = 1, 2$ put

$$\begin{aligned} \phi_i(t) &= z_1 \quad \text{for } t \in Y - G_i, \\ &= z_2 \quad \text{for } t \in G_i. \end{aligned}$$

Then $\phi_i \in \mathrm{Hom}_T(Y, Z)$, $\phi_1(y) \neq \phi_2(y)$, $\phi_1 f = \phi_2 f$, which is a contradiction.

3. *b*-embedding reflector.

3.1. DEFINITION. Let $F: T \to P$ be a reflector. F is called a *b-embedding reflector* iff $P \subset T_0$ and if $e_X(X)$ is a b-dense subset of $F(X)$ for all $X \in T$.

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[3] i_X denotes the identity on X.

3.2. Definition. A topological property P is called a *b-embedding* iff there exists a b-embedding reflector $F: T \to P$.

3.3. Definition. A topological property P is *b-closed-hereditary* if $A \in P$ whenever A is a b-closed subspace of some $Q \in P$.

3.4. Theorem. *A topological property P is b-embedding iff P is productive, b-closed-hereditary and $P \subset T_0$.*

Proof. Let B denote the category of all ordered triples (A, e, X), where A is a set, $X \in T_0$, e is a mapping from B into X and $be(A) = X$. A morphism $g: (A, e, X) \to (C, f, Y)$ is a mapping g from A into C for which there exists a continuous mapping $\bar{g} \in \operatorname{Hom}_T(X, Y)$ such that $\bar{g}e = fg$.

Let S be the category of all sets and mappings, K be the functor $B \to S$ for which $K(A, e, X) = A$ and $K(g) = g$. By means of the assertions of §2 we prove as in the third section of Kennison's paper [4] that K is a pullback stripping functor, whence we prove this theorem in the same manner as Theorems C and D are proved in [4].

Remark. This theorem can be proved easily also by means of the Theorem 3 in Hušek's paper [2] and by means of the assertion of §2 of this paper.

3.5. Theorem. *Let P be a topological property, which is a reflective subcategory of T. If $P \subset T_0$, $P \not\subset T_1$, then P is b-embedding. If $P \not\subset T_0$, then P is simple.*

Proof. According to 1.1 P is productive.

I. Let $P \subset T_0$, P non$\subset T_1$. Let F be a reflector: $T \to P$. Let $X \in P$, $Y \subset X$, $bY = Y$, i denote the identical embedding Y into X. Then there exists one morphism $j \in \operatorname{Hom}_T(F(Y), X)$ such that $je_Y = i$. Thus it follows that e_Y is a homeomorphism from Y into $F(Y)$. According to 2.8 $be_Y(Y) = F(Y)$ holds; from 2.5 we get $jF(Y) = Y$ and from 2.6 it follows $e_Y(Y) = F(Y)$; hence P is a b-embedding.

II. Let P non$\subset T_0$. Then there exist $Z \in T$, $z_1, z_2 \in Z$, $z_1 \neq z_2$, $z_1 \in \operatorname{cl}\{z_2\}$, $z_2 \in \operatorname{cl}\{z_1\}$. Then each mapping f of a space X into Z with the property $f(X) \subset \{z_1, z_2\}$ is continuous; thus it follows easily that P is simple.

4. **Remark.** Let $Q = \{q_1 \neq q_2\}$ denote a two-element set and $U = \{\{q_1\}, Q, \varnothing\}$. Then $(Q, U) \in T_0$. Let P' be the class of all topological spaces of the type $\prod Q_\iota$ ($\iota \in \mathscr{I}$), where $\mathscr{I}$ is a nonvoid set and $Q_\iota = (Q, U)$ for each $\iota \in \mathscr{I}$. Let P (resp. P^*) be the class of all b-closed (resp. closed) subspaces of a space from P' and of all spaces homeomorphic to these spaces.

4.1. *P (resp. P^*) is a topological property, which is productive and b-closed (resp. closed) hereditary.*

Proof. We can see clearly that P is a topological property, which is b-closed hereditary. From 2.7 it follows that P is productive. We can prove easily the assertion about the class P^*.

4.2. *Let $\varnothing \neq X$ be a subspace of $Y \in T_0$, let $Z \in P'$, $f \in \operatorname{Hom}_T(X, Z)$. Then there exists at least one $g \in \operatorname{Hom}_T(Y, Z)$, $g|X = f$.*

Proof. I. Let $Z=(Q, U)$. Then $f^{-1}(q_1)$ is an open set of X; therefore there exists an open set G of Y such that $G \cap X=f^{-1}(q_1)$. Put

$$\begin{aligned} g(y) &= q_1 \quad \text{for } y \in G, \\ &= q_2 \quad \text{for } y \in Y-G. \end{aligned}$$

Then $g \in \operatorname{Hom}_T(Y, Z)$ and $g|X=f$.

II. Let $Z=\prod Q_\iota$ $(\iota \in \mathscr{I} \neq \varnothing)$, $Q_\iota=(Q, U)$, p_ι be the projection $Z \to Q_\iota$. According to I for each $\iota \in \mathscr{I}$ there exists $g_\iota \in \operatorname{Hom}_T(Y, Q_\iota)$, $g_\iota|X=p_\iota f$. Then the mapping g defined $p_\iota g=g_\iota$ has the property sought.

4.3. *Let* $\varnothing \neq X \in P^*$. *Then there exists* $x_0 \in X$ *such that* x_0 *has only one neighborhood* X.

Proof. If $X \subset \prod Q_\iota$ $(\iota \in \mathscr{I} \neq \varnothing)$, $Q_\iota=(Q, U)$ and X is closed, then we shall put $x_0(\iota)=q_2$ for each $\iota \in \mathscr{I}$. Then $x_0 \in X$ and only one neighborhood of x_0 is X.

4.4. Let $R=\prod_{n=1}^{\infty} Q_n$, where $Q_n=(Q, U)$, $S=R-\{r_0\}$, where $r_0(n)=q_1$ for each positive integer n. Since each nonvoid open set of R contains the point r_0, it holds $b_R S=R$. From 4.2 and 2.5 R is the reflection of S in P. According to the Theorem 3.4 *P is the reflective full (replete) subcategory of T and P is neither simple nor identifying, nor embedding*, which are types mentioned in Kennison's paper [4].

Let $S^*=\{x \in R \mid$ there exists a positive integer n such that $x(n)=q_1\}$. S^* is an open set of R and according to Corollary of 2.2 $b_R S^*=S^*$. From 4.3 $S^* \notin P^*$. Therefore according to the Theorem 3.5 P^* is not a reflective subcategory of T. From this the *b-closed-hereditary property in Theorem* 3.4 *cannot be replaced by the closed-hereditary property.*

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University of Brno,
Brno, Czechoslovakia